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# On the Nash equilibrium of partial cooperative games

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## 1 Introduction

In [4] a class of partial cooperative games with perfect information (PCGPI) is defined. PCGPI proceeds on a tree  $K(x_0)$  of a finite non-cooperative game in extensive form with perfect information and without chance moves  $\Gamma = \langle K(x_0), P, h \rangle$ . Here,  $x_0$  is the origin of  $K(x_0)$ ;  $P$  denotes the player partition  $P_1, \dots, P_i, \dots, P_n, P_{n+1}$ , where  $P_i, i \in N$ , is the set of decision points of player  $i$ , and  $P_{n+1}$  is the set of the endpoints;  $h : P_{n+1} \rightarrow R_+^n$  is the terminal payoff function. Denote the player set by  $N = \{1, \dots, n\}$ . In PCGPI for each player  $i$  a set of points called the cooperative region is given. (In general case the cooperative region may be empty.) During the game, in a decision point  $x \in P_i$  player  $i$  is purposed to use an individually rational behavior if  $x$  is not in his cooperative region. But, if  $x$  lies in the cooperative region of player  $i$ , then in  $x$  he forms a coalition involving all players whose cooperative regions contain  $x$  also.

Formalization of the concept of the players' cooperative region may be realized by various approaches. In [4] a timing interpretation of the cooperative region is considered. It is supposed that  $K(x_0)$  has the following information structure:

1. For any evolution of the game players make decisions in accordance with their index order, i.e., in the point  $x_0$  the decision is made by player 1, in the immediate successors of  $x_0$  the decision is made by player 2 and so on until player  $n$ . After player  $n$  the decision is again made by player 1 and etc.
2. Each path has the same length.

For the given game tree, we shall say that a *stage* is the  $n$  sequential moves, where the first move is made by player 1. Let the length of  $K(x_0)$  be  $T + 1$  stages.

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In PCGPI a vector  $s = (s_1, \dots, s_i, \dots, s_n)$ ,  $s_i \in \bar{L} = \{0, 1, \dots, T, T+1\}$ , is given. The component  $s_i$  denotes the length of the player  $i$ 's cooperative activity. If  $s_i = 0$ , then during the game player  $i$  plays non-cooperatively. If  $s_i > 0$ , then starting from the initial stage 0 until the stage  $T - s_i$  player  $i$  plays non-cooperatively, and since the stage  $t_i = T - s_i + 1$  until the end of the game player  $i$  is ready to cooperate with anybody. The given PCGPI is denoted by  $\Gamma_s(x_0)$ .

Suppose that  $\{x_0, \dots, \bar{x}\}$  is the path realized in  $\Gamma_s(x_0)$ . Let  $S_s = \{i \in N | s_i > 0\}$  be a coalition formed to the end of  $\Gamma_s(x_0)$ . If  $i \notin S$ , then the payoff of player  $i$  is defined by the terminal payoff function  $h$  and equals  $h_i(\bar{x})$ . If  $i \in S_s$ , then the payoff of player  $i$  is defined by the Shapley value  $\alpha(s)$  of the payoff of the coalition  $S_s$ , i.e.,

$$\sum_{j \in S_s} \alpha_j(s) = \sum_{j \in S_s} h_j(\bar{x})$$

It is considered that the purpose of a player in  $\Gamma_s(x_0)$  is maximizing his payoff within the restrictions given by  $s$ .

Let  $L = \prod_{i \in N} \bar{L}$  be the set of all vectors  $s$  that can be defined for  $K(x_0)$ . In [4] an approach to find the players' optimal behavior in  $\Gamma_s(x_0)$ ,  $s \in L$ , is proposed. The scheme of construction of a path  $\Phi_s(x_0) = \{x_0, \dots, \phi_s(x_0)\}$ ,  $\phi_s(x_0) \in P_{n+1}$ , which is realized in  $\Gamma_s(x_0)$  when players keep on their optimal behavior, is defined. The payoff-vector  $r(s) = (r_1(s), \dots, r_n(s))$ ,

$$r_i(s) = \begin{cases} h_i(\phi_s(x_0)), & \text{if } s_i = 0 \\ \alpha_i(s), & \text{if } s_i > 0, \end{cases} \quad i \in N$$

related to  $\Phi_s(x_0)$  is called the *value* of  $\Gamma_s(x_0)$ .

In  $\Gamma_s(x_0)$  the vector  $s$  is not regulated by players. In this paper we consider a generalization of  $\Gamma_s(x_0)$ , where players form a vector  $s \in L$  themselves.

## 2 Model.

On the tree  $K(x_0)$  consider a new game  $\Gamma_L(x_0)$ . In pre-play communications of  $\Gamma_L(x_0)$  players form a vector  $s \in L$ . Then, players play in accordance with the vector  $s$ . Hence,  $\Gamma_L(x_0)$  evolves along the optimal path  $\Phi_s(x_0)$  and players get payoffs defined by the value  $r(s)$ . It is supposed that in  $\Gamma_L(x_0)$  each player tries to maximize his own payoff.

**Definition.** A vector  $s^* \in L$  is called the *Nash equilibrium* of  $\Gamma_L(x_0)$  if for all  $s_i \in \bar{L}$  and  $i \in N$  there is

$$r_i(s^*) \geq r_i(s^* | s_i), \quad (2.1)$$

where  $s^* | s_i = (s_1^*, \dots, s_{i-1}^*, s_i, s_{i+1}^*, \dots, s_n^*)$ .

**Theorem.** The Nash equilibrium in  $\Gamma_L(x_0)$  always exists.

*Proof.* We prove the theorem if propose the Nash equilibrium construction method for  $\Gamma_L(x_0)$ .

Knowing the formed vector  $s$  we know the path  $\Phi_s(x_0)$  of the game evolution and players' payoffs  $r(s)$ . Therefore, the set  $\bar{L}$  may be considered as the set of the player's strategies in  $\Gamma_L(x_0)$ . For each player  $i$  and his decision point  $x \in P_i$ , if player  $i$  cooperates in  $x$  or not that is all we need to know.

Basing on  $K(x_0)$ , define an auxiliary binary tree  $\bar{K}(x_0)$ . The length of  $\bar{K}(x_0)$  is  $T + 1$  stages (the definition of a stage is given in section 1). For each decision point  $x$ , we shall call the branches going out from  $x$  by *Left* and *Right* respectively. We shall consider that if player  $i$  does not cooperate in a stage  $t$ , then on  $\bar{K}(x_0)$  player  $i$  has to go Left in his decision points in the stage  $t$ . Otherwise, if player  $i$  cooperates in the stage  $t$ , then on  $\bar{K}(x_0)$  player  $i$  has to go Right in his decision points in the stage  $t$ . For the given relation between the rules of  $\Gamma_L(x_0)$  and  $\bar{K}(x_0)$  to be one-to-one, we suppose that  $\bar{K}(x_0)$  satisfies the following condition.

Let  $x_r$  and  $x_l$  be immediate successors of a decision point  $x$ . Assume that  $x_r$  related to the decision Right in  $x$ , and  $x_l$  related to the decision Left in  $x$ . Then,

$$K(x_r) \cap P_i = \emptyset \quad (2.2)$$

for each player  $i \in N$  and his decision point  $x \in P_i$ . Here,  $K(x_r)$  denotes the subtree with the initial point  $x_r$ .

Let  $\bar{P}_1, \dots, \bar{P}_n, \bar{P}_{n+1}$  be the player partition on  $\bar{K}(x_0)$ , where  $\bar{P}_{n+1}$  is the set of endpoints. By the condition (2.2) there is one-to-one correspondence between the sets  $L$  and  $\bar{P}_{n+1}$ . Define a payoff function  $\bar{h} : \bar{P}_{n+1} \rightarrow R_+^n$  by

$$\bar{h}(\hat{x}) = r(s), \quad \hat{x} \in \bar{P}_{n+1} \quad (2.3)$$

where  $\hat{x}$  related to  $s$ . Consider a non-cooperative game  $\bar{\Gamma} = \langle \bar{K}(x_0), \bar{P}, \bar{h} \rangle$ . Let  $\pi = (\pi_1, \dots, \pi_n)$  denote a situation in  $\bar{\Gamma}$ , where  $\pi_i$ ,  $i \in N$ , is a player  $i$ 's strategy. Denote the set of all situations in  $\bar{\Gamma}$  by  $\Pi$ . Suppose that  $\pi^*$  is the Nash equilibrium in  $\bar{\Gamma}$ . From the construction of the game  $\bar{\Gamma}$  it follows that there is one-to-one correspondence between  $\Pi$  and  $L$ . Hence, the vector  $s^*$  related to  $\pi^*$  satisfies the definition of the Nash equilibrium in  $\Gamma_L(x_0)$ .  $\square$

**Remark.** During the theorem proof a construction method of the Nash equilibrium in  $\Gamma_L(x_0)$  was proposed.

**Example.** Consider a three person non-cooperative game  $\Gamma$  with the game tree  $K(x_0)$  given in Figure 1.  $N = \{1, 2, 3\}$ . The player 1's decision points are denoted by single circle, player 2's — by double circle and player 3's — by triple circle. The vectors at the endpoints are the terminal payoffs of players, with the first components being the payoff of player 1 and so on. There are two stages in  $\Gamma$ . The initial stage starts in  $x_0$ . The stage 1 starts in  $x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}$ .  $\bar{L} = \{0, 1, 2\}$ . For each  $s \in L$  consider the game  $\Gamma_s(x_0)$  and find the value  $r(s)$ . All possible values  $r(s)$ ,  $s \in L$ , are given in Table 1.

Find the Nash equilibrium  $s^*$  of the game  $\Gamma_L(x_0)$ . Construct the tree  $\bar{K}(x_0)$  (see in Figure 2) which satisfies the condition (2.2).

We shall say that player 1 goes Up in  $x_0 \in \bar{K}(x_0)$ , if he cooperates in  $\Gamma_L(x_0)$  since the initial stage. Player 1 goes Down in  $x_0 \in \bar{K}(x_0)$ , if he does not cooperate in  $\Gamma_L(x_0)$  in the initial stage. In  $x_{11}, x_{12}, x_{13}$  and  $x_{14}$  of  $\bar{K}(x_0)$  player 1 goes Up, if he cooperates in  $\Gamma_L(x_0)$  since the stage 1. If player 1 does not cooperate in  $\Gamma_L(x_0)$ , then he goes Down in  $x_{11}, x_{12}, x_{13}$  and  $x_{14}$  of  $\bar{K}(x_0)$ .

Player 2 goes Up in  $x_1, x_2$  of  $\bar{K}(x_0)$ , if he cooperates in  $\Gamma_L(x_0)$  since the initial stage  $x_1, x_2$  of  $\bar{K}(x_0)$ . If player 2 does not cooperate in  $\Gamma_L(x_0)$  in the initial stage, then he goes Down

in  $x_1, x_2$  of  $\bar{K}(x_0)$ . Player 2 goes Up in  $x_9, x_{10}, x_{25}, x_{26}, x_{27}, x_{28}$  of  $\bar{K}(x_0)$ , if he cooperates in  $\Gamma_L(x_0)$  since the stage 1. If player 2 does not cooperate in  $\Gamma_L(x_0)$ , then he goes Down in  $x_9, x_{10}, x_{25}, x_{26}, x_{27}, x_{28}$  of  $\bar{K}(x_0)$ .

Player 3 goes Up in  $x_3, x_4, x_5, x_6$  of  $\bar{K}(x_0)$ , if he cooperates in  $\Gamma_L(x_0)$  since the initial stage. If player 3 does not cooperate in  $\Gamma_L(x_0)$  in the initial stage, then he goes Down in  $x_3, x_4, x_5, x_6$  of  $\bar{K}(x_0)$ . Player 3 goes Up in  $x_8, x_{19}, x_{20}, x_{23}, x_{24}, x_{41}, x_{42}, x_{43}, x_{44}$  of  $\bar{K}(x_0)$ , if he cooperates in  $\Gamma_L(x_0)$  since the stage 1. If player 3 does not cooperate in  $\Gamma_L(x_0)$ , then he goes Down in  $x_8, x_{19}, x_{20}, x_{23}, x_{24}, x_{41}, x_{42}, x_{43}, x_{44}$  of  $\bar{K}(x_0)$ .

Using the given interpretation of players' behavior, we put the values  $r(s)$ ,  $s \in L$ , at the endpoints of  $\bar{K}(x_0)$ . Define the non-cooperative game  $\bar{\Gamma}$  on  $\bar{K}(x_0)$  and find the Nash equilibrium of  $\bar{\Gamma}$ .

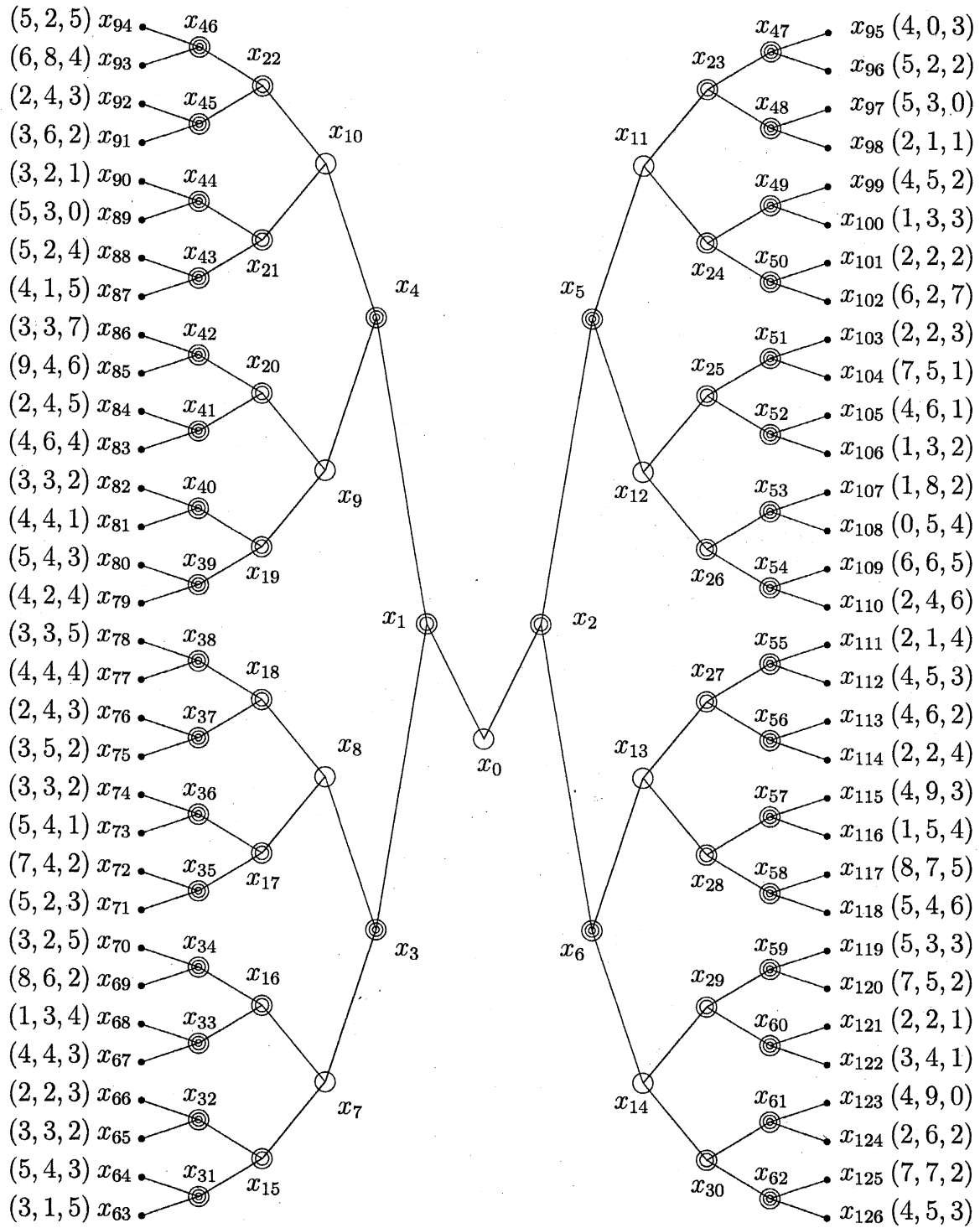
There are tree Nash equilibrium in  $\bar{\Gamma}$ . The trajectories related to the Nash equilibrium situations are  $\{x_0, \dots, x_{23}\}$ ,  $\{x_0, \dots, x_{35}\}$  and  $\{x_0, \dots, x_{45}\}$ . Hence, the Nash equilibriums in  $\Gamma_L(x_0)$  are  $(0, 2, 1)$ ,  $(0, 1, 2)$  and  $(0, 1, 1)$ . For all cases players get payoffs  $(9, 4\frac{1}{2}, 5\frac{1}{2})$ . We can see that for player 1 it is optimal (in the sense of the Nash equilibrium) not to cooperate in  $\Gamma_L(x_0)$ . Note, that if all players cooperate since the start of  $\Gamma_L(x_0)$ , then we have a usual cooperative game on  $K(x_0)$ . In this case, the Shapley value is  $(7, 7, 6)$ .

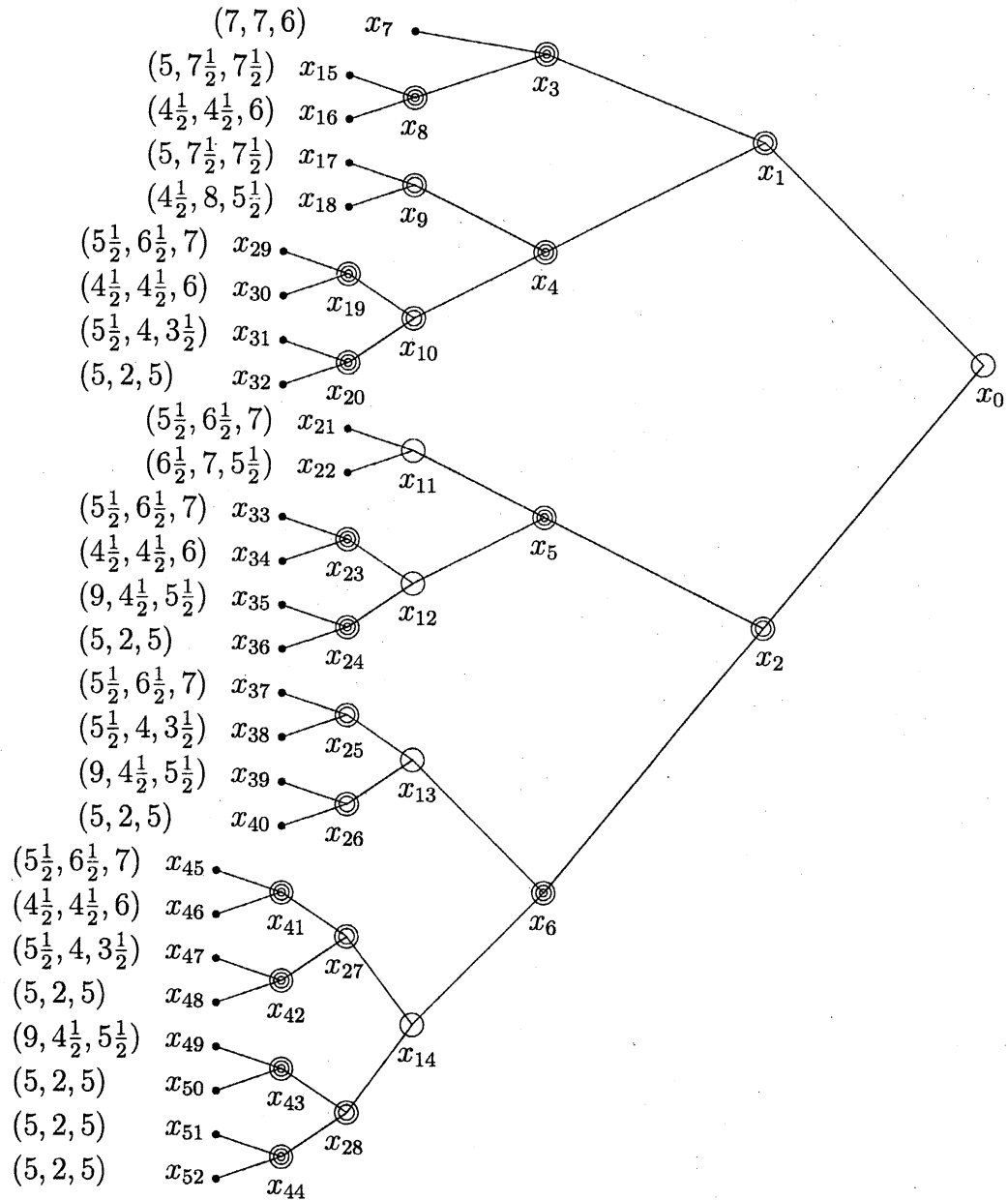
$s = (s_1, s_2, s_3)$	$r(s)$	$s = (s_1, s_2, s_3)$	$r(s)$	$s = (s_1, s_2, s_3)$	$r(s)$
$(0, 0, 0)$	$(5, 2, 5)$	$(2, 0, 0)$	$(5, 2, 5)$	$(1, 0, 2)$	$(5\frac{1}{2}, 4, 3\frac{1}{2})$
$(1, 0, 0)$	$(5, 2, 5)$	$(0, 2, 0)$	$(5, 2, 5)$	$(2, 2, 0)$	$(4\frac{1}{2}, 4\frac{1}{2}, 6)$
$(0, 1, 0)$	$(5, 2, 5)$	$(0, 0, 2)$	$(5, 2, 5)$	$(0, 2, 2)$	$(6\frac{1}{2}, 7, 5\frac{1}{2})$
$(0, 0, 1)$	$(5, 2, 5)$	$(2, 1, 0)$	$(4\frac{1}{2}, 4\frac{1}{2}, 6)$	$(2, 0, 2)$	$(4\frac{1}{2}, 8, 5\frac{1}{2})$
$(1, 1, 0)$	$(4\frac{1}{2}, 4\frac{1}{2}, 6)$	$(1, 2, 0)$	$(4\frac{1}{2}, 4\frac{1}{2}, 6)$	$(2, 2, 1)$	$(5, 7\frac{1}{2}, 7\frac{1}{2})$
$(1, 0, 1)$	$(5\frac{1}{2}, 4, 3\frac{1}{2})$	$(0, 2, 1)$	$(9, 4\frac{1}{2}, 5\frac{1}{2})$	$(1, 2, 2)$	$(5\frac{1}{2}, 6\frac{1}{2}, 7)$
$(0, 1, 1)$	$(9, 4\frac{1}{2}, 5\frac{1}{2})$	$(0, 1, 2)$	$(9, 4\frac{1}{2}, 5\frac{1}{2})$	$(2, 1, 2)$	$(5, 7\frac{1}{2}, 7\frac{1}{2})$
$(1, 1, 1)$	$(5\frac{1}{2}, 6\frac{1}{2}, 7)$	$(2, 0, 1)$	$(5\frac{1}{2}, 4, 3\frac{1}{2})$	$(2, 2, 2)$	$(7, 7, 6)$
$(2, 1, 1)$	$(5\frac{1}{2}, 6\frac{1}{2}, 7)$	$(1, 2, 1)$	$(5\frac{1}{2}, 6\frac{1}{2}, 7)$	$(1, 1, 2)$	$(5\frac{1}{2}, 6\frac{1}{2}, 7)$

Table 1: Players' payoffs

We supposed that each player use the following criteria when he make decision in  $\bar{\Gamma}$ .

- 1) to maximize own payoff;
- 2) if criterion 1 is fulfilled, then to maximize the common payoff of all players;
- 3) if criteria 1, 2 are fulfilled, then to maximize the payoff of player 1 (if the player is not player 1);
- 4) if criteria 1,  $\dots$ , 3 are fulfilled, then to maximize the payoff of player 2 (if the player is not player 2) and so on;
- $n + 2$ ) if criteria 1,  $\dots$ ,  $n + 1$  are fulfilled, then to maximize the payoff of player  $n$  (if the player is not player  $n$ );
- $n + 3$ ) if criteria 1,  $\dots$ ,  $n + 2$  are fulfilled, then to choose any of the remain strategies.

Figure 1: The game tree  $K(x_0)$

Figure 2: The game tree  $\bar{K}(x_0)$

## References

- [1] Hsiao C.R. and Raghavan T.E.S., *The Shapley value for multi-choice games (I)*, Games and Economic Behavior **5** (1993), 240–256.
- [2] Neumann J. von and Morgenstern O., *Theory of games and Economic Behavior*, Princeton University Press, Princeton, New Jersey, (1944)
- [3] Nash J.F., *Non-cooperative games*, Ann. of Math. **54** (1951), 286–295.
- [4] Petrosjan L.A., Ayoshin D.A., Tanaka T., *Construction of a Time Consistent Core in Multi-Choice Multistage Games*, RIMS – 97, Symposium “Decision Theory and Its Related Fields” (1997).
- [5] Shapley L.S., *A value for  $n$ -person games*, Ann. of Math. **28** (1953), 307–317.
- [6] Vorobjev N.N., *Game Theory Lectures for Economists and System Scientists*, Springer-Verlag, New-York (1977).